# ON THE PROBLEM OF STABILITY OF EQUILIBRIUM POSITIONS OF HAMILTONIAN SYSTEMS

PMM Vol. 34, №6, 1970, pp. 997-1004
A. P. MARKEEV
(Moscow)
(Received April 20, 1970)

The stability of equilibrium positions of Hamiltonian systems with one or two degrees of freedom in the presence of resonance is investigated. The conditions of instability, as well as those of Liapunov stability for cases in which only formal stability is known are derived.

**1.** Let

$$\frac{dx}{dt} = \frac{\partial H}{\partial y}, \qquad \frac{dy}{dt} = -\frac{\partial H}{\partial x} \tag{1.1}$$

be a canonical system where the Hamiltonian $H$ is analytic with respect to $x$, $y$ in the neighborhood of the origin of coordinates

$$H = \sum_{k=2} H_k(x, y, t), \quad H_k = \sum_{\nu_1+\nu_2=k} a_{\nu_1\nu_2}(t)\, x_{\nu_1} y^{\nu_2}, \quad a_{\nu_1\nu_2}(t+2\pi) \equiv a_{\nu_1\nu_2}(t)$$

Let us assume that the linearized system is stable and the characteristic indexes $\pm i\lambda$ are such, that $k\lambda$ is not an integer for $k = 1, 2, \ldots, 2n$. Then with a suitable choice of variables $x$, $y$ the Hamiltonian can be written in the form[1]

$$H = \lambda r + c_2 r^2 + \ldots + c_n r^n + H'(x, y, t) \qquad (2r = x^2 + y^2) \tag{1.2}$$

Here $H' = O(r^{n+1/2})$ is an analytic function of $x$, $y$. If among the constants $c_2$, $c_3$, ..., $c_n$ there is one distinct from zero, the equilibrium position $x = y = 0$ is stable [2, 3].

If however $k\lambda$ is an integer, the Hamiltonian cannot be, generally speaking, reduced to the form (1.2), and the equilibrium position may be unstable. In this paper we consider the problem of stability in the presence of resonance, when for $k \geqslant 3$ the quantity $k\lambda$ is an integer.

The fundamental result of this investigation is the confirmation of stability (with certain inequality satisfied) in cases of resonance of an even order ($k$ is an even number). Confirmation of instability in one form or another was obtained earlier in [4-8].

The stability of equilibrium position of an autonomous Hamilton system with two degrees of freedom, when the ratio of frequencies of the linearized system is equal to three, is considered in Sect. 6.

**2.** Let us investigate the stability of the equilibrium position of system (1.1) in the region of stability of the linear approximation system. This implies that $2\lambda$ is not an integer. Further calculations require to find a real, canonical, $2\pi$-periodic with respect to $t$, and linear with respect to $x$, $y$, transformation of the Hamiltonian by which its quadratic part becomes of the form $H_2 = {}^1/_2\, \lambda\, (q^2 + p^2)$. We will show how to obtain this transformation.

The linearized system (1.1) has two linearly independent solutions

$$\alpha_j = \varphi_j(t)\, e^{i\lambda_j t}, \quad \beta_j = \psi_j(t)\, e^{i\lambda_j t} \qquad (j = 1, 2) \tag{2.1}$$

where $\lambda_1 = -\lambda_2 = -\lambda$, and the periodic functions $\varphi_j, \psi_j$ satisfy differential equations (2.2)

$$d\varphi_j/dt = -i\lambda_j\varphi_j + a_{11}\varphi_j + 2a_{02}\psi_j, \qquad d\psi_j/dt = -i\lambda_j\psi_j - 2a_{20}\varphi_j - a_{11}\psi_j$$

If the initial values $\varphi_1$, $\psi_1$ are complex conjugates of the initial values $\varphi_2$ and $\psi_2$, then, by virtue of the homogeneity of system (2.2), these functions are complex congugates for all $t$. Hence it can be set

$$\varphi_1 = z_1 + iz_2, \quad \psi_1 = z_3 + iz_4, \quad \varphi_2 = \overline{\varphi}_1, \quad \psi_2 = \overline{\psi}_1$$

where $z_j$ are real periodic functions of $t$. According to (2.2) they satisfy the following system of equations:

$$\begin{aligned} dz_1/dt &= -\lambda z_2 + a_{11}z_1 + 2a_{02}z_3, & dz_2/dt &= \lambda z_1 + a_{11}z_2 + 2a_{02}z_4 \\ dz_3/dt &= -\lambda z_4 - 2a_{20}z_1 - a_{11}z_3, & dz_4/dt &= \lambda z_3 - 2a_{20}z_2 - a_{11}z_4 \end{aligned} \tag{2.3}$$

It is readily seen that the linearized system (1.1) has two independent integrals

$$(u + iv)e^{-i\lambda t}, \qquad (u - iv)e^{i\lambda t}\,(u = z_3x - z_1y,\ v = z_4x - z_2y) \tag{2.4}$$

We introduce new variables $q$ and $p$ defined by functions $q = v$ and $p = u$. This transformation is canonical, since functions $z_j$ satisfy the relation

$$z_2z_3 - z_1z_4 = \text{const} \tag{2.5}$$

which can be readily verified by a direct check.

Let us choose the initial values of functions $z_j$ so that the initial values of functions $\varphi_1$, $\psi_1$ and $\varphi_2$, $\psi_2$ be complex conjugates, and the constant in (2.5) be equal unity.

Let us denote by $x_j(t), y_j(t)$ $(j = 1, 2)$ the solutions of the linearized system (1.1) which satisfy conditions

$$x_1(0) = y_2(0) = 1, \qquad x_2(0) = y_1(0) = 0$$

The initial values of functions $\varphi_j$, $\psi_j$ are then found from the system of equations

$$\begin{aligned} &[x_1(2\pi) - e^{i2\pi\lambda_j}]\ \varphi_j(0) + x_2(2\pi)\ \psi_j(0) = 0 \\ &y_1(2\pi)\ \ \varphi_j(0) + [y_2(2\pi) - e^{i2\pi\lambda_j}]\ \psi_j(0) = 0 \end{aligned} \tag{2.6}$$

The determinants of these systems are equal to zero, since the $e^{i2\pi\lambda_j}$ are the multipliers of the linearized system (1.1). The solutions of system (2.6) can be written as

$$\varphi_j(0) = -x_2(2\pi)\ c_j, \qquad \psi_j(0) = [x_1(2\pi) - e^{i2\pi\lambda_j}]c_j \tag{2.7}$$

where $c_j$ are arbitrary constants. Let these be real and equal to $c$. Then $\varphi_1(0) = \bar{\varphi}_2(0)$, $\psi_1(0) = \overline{\psi}_2(0)$.

From (2.7) we obtain the initial values of function $z_j$

$$\begin{aligned} z_1(0) &= -x_2(2\pi)\ c, & z_2(0) &= 0 \\ z_3(0) &= [x_1(2\pi) - \cos 2\pi\lambda]\ c, & z_4(0) &= c\cdot\sin 2\pi\lambda \end{aligned} \tag{2.8}$$

Setting the constant in (2.5) equal unity, we obtain for the determination of $c$ the condition

$$c^2x_2(2\pi)\sin 2\pi\lambda = 1 \tag{2.9}$$

The quantity $x_2(2\pi)\sin 2\pi\lambda \neq 0$, since the stability is investigated in the region of stability of the linearized system (1.1). By choosing the sign of $\lambda$ (which so far has not been defined) this quantity can be made positive. Hence Eq. (2. 9) has always a real solution for $c$.

The sought canonical transformation has thus been found, and the Hamiltonian in terms of variables $q$, $p$ is

$$H = {}^1/_2\lambda\,(q^2 + p^2) + \sum_{k=3}^{\infty} H_k\,(q, p, t) \tag{2.10}$$

where

$$H_k = \sum_{\nu_1+\nu_2=k} h_{\nu_1\nu_2}(t)\, q^{\nu_1}p^{\nu_2} \equiv \sum_{\nu_1+\nu_2=k} a_{\nu_1\nu_2}(t)\,(z_2 p - z_1 q)^{\nu_1}(z_4 p - z_3 q)^{\nu_2}$$

**3.** To investigate the stability we further transform the Hamiltonian (2.10). Using the periodic generating function of $t$ of period $2\pi$, we introduce the canonical variables $q^*$, $p^*$

$$S = qp^* + S^{(3)} \equiv qp^* + \sum_{\nu_1+\nu_2=3} s_{\nu_1\nu_2}(t)\, q^{\nu_1}p^{*\nu_2}$$

It is not difficult to show that, if $3\lambda$ is not an integer, the third power terms in the new Hamiltonian $H^*$ $(q^*, p^*, t)$ can be completely supressed. For this the $2\pi$-periodic functions $s_{\nu_1\nu_2}(t)$ must be such that

$$s_{30} = 2\,(u_{30}' + u_{21}'), \qquad s_{03} = 2\,(v_{30}' - v_{21}')$$

$$s_{12} = 2\,(u_{21}' - 3u_{30}'), \qquad s_{21} = -2\,(3v_{30}' + v_{21}') \tag{3.1}$$

$$u_{\nu_1\nu_2}' = f(t)\sin\lambda(\nu_2 - \nu_1)\,t + g(t)\cos\lambda(\nu_2 - \nu_1)\,t$$

$$v_{\nu_1\nu_2}' = f(t)\cos\lambda(\nu_2 - \nu_1)\,t - g(t)\sin\lambda(\nu_2 - \nu_1)\,t$$

$$f(t) = {}^1/_2 \operatorname{ctg}\pi\lambda(\nu_2 - \nu_1)\; J_1(2\pi) + {}^1/_2\, J_2(2\pi) - J_2(t) \tag{3.2}$$

$$g(t) = -{}^1/_2 \operatorname{ctg}\pi\lambda(\nu_2 - \nu_1)\; J_2(2\pi) + {}^1/_2\, J_1(2\pi) - J_1(t)$$

$$J_1(t) = \int_0^t [u_{\nu_1\nu_2}''(x)\cos\lambda(\nu_2 - \nu_1)\,x - v_{\nu_1\nu_2}''(x)\sin\lambda(\nu_2 - \nu_1)\,x]\,dx$$

$$J_2(t) = \int_0^t [u_{\nu_1\nu_2}''(x)\sin\lambda(\nu_2 - \nu_1)\,x + v_{\nu_1\nu_2}''(x)\cos\lambda(\nu_2 - \nu_1)\,x]\,dx$$

$$u''_{30} = {}^1/_8\,(h_{30} - h_{12}), \qquad v''_{30} = {}^1/_8\,(h_{03} - h_{21}) \tag{3.3}$$

$$u''_{21} = {}^1/_8\,(3h_{30} + h_{12}), \qquad v''_{21} = -{}^1/_8\,(3h_{03} + h_{21})$$

With such choice of $S^{(3)}$ the fourth power terms in $H^*$ are computed by the formula

$$H_4^*(q, p^*, t) = H_4 + {}^1/_2\lambda\left[\left(\frac{\partial S^{(3)}}{\partial q}\right)^2 - \left(\frac{\partial S^{(3)}}{\partial p^*}\right)^2\right] + \frac{\partial H_3}{\partial p^*}\frac{\partial S^{(3)}}{\partial q} \tag{3.4}$$

When $3\lambda = m$ ($m$ is an integer), it is not possible to completely suppress $H_3^*$. In this case the Hamiltonian $H^*$ can be reduced to

$$H^* = {}^1/_2\lambda\,(q^{*2} + p^{*2}) + 2u_{30}^*\,(q^{*3} - 3q^*p^{*2}) + 2v_{30}^*\,(p^{*3} - 3p^*q^{*2}) + H'(q^*, p^*, t) \tag{3.5}$$

where

$$u_{30}^* = x_{30}\cos mt - y_{30}\sin mt, \quad v_{30}^* = x_{30}\sin mt + y_{30}\cos mt$$

$$x_{30} = \frac{1}{2\pi}\int_0^{2\pi} (u_{30}''\cos mt + v_{30}''\sin mt)\,dt \tag{3.6}$$

$$y_{30} = \frac{1}{2\pi}\int_0^{2\pi} (v_{30}''\cos mt - u_{30}''\sin mt)\,dt$$

The function $H'$ has a period of $2\pi$ with respect to $t$, and $H' = O((|q| + |p|)^4)$.

Theorem 3.1. If $x_{30}^2 + y_{30}^2 \neq 0$, the equilibrium position is unstable.

Proof. Let us make a substitute of variables

$$q^* = \sqrt{2r}\sin(\lambda t + \varphi - \theta), \qquad p^* = \sqrt{2r}\cos(\lambda t + \varphi - \theta)$$

$$\sin 3\theta = x_{30}(x_{30}^2 + y_{30}^2)^{-1/2}, \qquad \cos 3\theta = y_{30}(x_{30}^2 + y_{30}^2)^{-1/2}$$

In terms of variables $r$, $\varphi$ the Hamiltonian (3.5) becomes

$$H = 4\sqrt{2(x_{30}^2 + y_{30}^2)}\, r\sqrt{r}\cos 3\varphi + O(r^2) \tag{3.7}$$

Let us consider the Liapunov function

$$V = r\sqrt{r}\sin 3\varphi \tag{3.8}$$

Its derivative by virtue of the equation of motion with the Hamiltonian (3.7) is

$$dV / dt = 18\sqrt{2(x_{30}^2 + y_{30}^2)}\, r^2 + O(r^{5/2}) \tag{3.9}$$

Since $V$ is an alternating function, and $dV / dt$ is positive definite in the neighborhood of the coordinate origin, the equilibrium position is, according to Liapunov's instability theorem [9], unstable.

**4.** If $3\lambda$ is not an integer, the Hamiltonian expressed in terms of variables $q^*$, $p^*$ is of the form

$$H^* = H_2^* + H_4^* + \ldots$$

where $H_4^*$ is computed by the formulas (3.1)–(3.4). Let $4\lambda = m$. Substituting the variables $q^*, p^* \to q^\circ, p^\circ$ with the generating function $S = q^*p^\circ + S^{(4)}$, we can simplify the fourth degree terms of the new Hamiltonian which, as shown by calculations, is of the form

$$H = {}^1/_2\lambda(q^{\circ 2} + p^{\circ 2}) + {}^1/_4 c_2 (q^{\circ 2} + p^{\circ 2})^2 + u_{40}^*(q^{\circ 4} - 6q^{\circ 2}p^{\circ 2} + p^{\circ 4}) -$$
$$- 4v_{40}^* q^\circ p^\circ (q^{\circ 2} - p^{\circ 2}) + H^\circ(q^\circ, p^\circ, t) \tag{4.1}$$

where function $H^\circ = O((|q^\circ| + |p^\circ|)^5)$ has a period of $2\pi$ with respect to $t$. The following notation was introduced in (4.1)

$$c_2 = \frac{1}{4\pi}\int_0^{2\pi} (3h_{40}^* + h_{22}^* + 3h_{04}^*)\, dt$$

$$u_{40}^* = x_{40}\cos mt - y_{40}\sin mt, \qquad v_{40}^* = y_{40}\cos mt + x_{40}\sin mt$$

$$x_{40} = \frac{1}{2\pi}\int_0^{2\pi} (u_{40}''\cos mt + v_{40}''\sin mt)\, dt$$

$$y_{40} = \frac{1}{2\pi}\int_0^{2\pi} (v_{40}''\cos mt - u_{40}''\sin mt)\, dt \tag{4.2}$$

$$u_{40}'' = {}^1/_8\,(h_{40}^* - h_{22}^* + h_{04}^*), \qquad v_{40}'' = {}^1/_8\,(h_{13}^* - h_{31}^*)$$

If $x_{40}$ and $y_{40}$ are not simultaneously zero, we can pass to the variables $r$, $\varphi$ by using formulas similar to those of the previous Section. We obtain

$$H = r^2(c_2 + b_2\cos 4\varphi) + H''(r, \varphi, t) \tag{4.3}$$

where $b_2 = 4\sqrt{x_{40}^2 + y_{40}^2}$, and the function $H'' = O(r^{5/2})$ and is periodic with respect to $\varphi$ and $t$ with periods $2\pi$ and $8\pi$, respectively.

Theorem 4.1. If $|c_2| < b_2$, the equilibrium position is unstable, if $|c_2| > b_2$, the Liapunov stability holds.

Proof. To prove the first statement of the Theorem we consider the Liapunov function

$$V = r^3 \sin 4\varphi \tag{4.4}$$

The function $V$ is alternating in the neighborhood of the coordinate origin. For the derivative we obtain the expression

$$dV / dt = 8r^3 (l_2 + c_2 \cos 4\varphi) + O(r^{7/2}) \tag{4.5}$$

When the inequality $|c_2| < b_2$ is satisfied, function (4.5) is positive definite in a sufficiently small neighborhood of the coordinate origin. Hence, the equilibrium position is unstable.

Now let $|c_2| > b_2$. It is not difficult to show that in this case in the system with the Hamiltonian $h = r^2 (c_2 + b_2 \cos 4\varphi)$, $r$ is a periodic, and $\varphi$ a monotonic function of $t$.

Let us make the canonical transformation, reducing $h$ to variables: action $I$-angle $W$ [10]. These variables are related to $r$ and $\varphi$ by formulas

$$r = \frac{\partial S}{\partial \varphi}, \qquad W = \frac{\partial S}{\partial I}, \qquad S(I, \varphi) = \int_0^\varphi r\, d\varphi \tag{4.6}$$

Here $S$ is the generating function. The integral in (4.6) is computed for the condition

$$r^2 (c_2 + b_2 \cos 4\varphi) = h \tag{4.7}$$

Here $h = h(I)$ is an inverse function of

$$I(h) = \frac{1}{2\pi} \int_0^{2\pi} r\, d\varphi \tag{4.8}$$

Here $r$ denotes the function $r(\varphi, h)$, which is obtained from (4.7).

We note that the signs of coefficients $c_2$ and $b_2$ in (4.3) can be assumed to be the same. In fact, if this were not so, a change to the new angular variable $\varphi - 1/4\, \pi$ would yield a Hamiltonian in which the signs of these coefficients would be the same. We introduce the notation $k^2 = 2b_2 / (b_2 + c_2)$. By virtue of the conditions of Theorem 4.1 the inequalities $0 \leqslant \kappa^2 < 1$ are satisfied. After simple calculations, from (4.6)–(4.8) we obtain

$$S = \frac{\pi I}{4K(k)} F(2\varphi, k) \tag{4.9}$$

where $K$ and $F$ are complete and incomplete elliptic integrals of the first kind and $k$ is their modulus. From (4.6) and (4.9) we find the relationship between the new and the old variables

$$r = \pi I \left[2K(k) \operatorname{dn} \frac{4K(k)}{\pi} W\right]^{-1}, \qquad \varphi = \frac{1}{2} \operatorname{am} \frac{4K(k)}{\pi} W \tag{4.10}$$

A further substitution of variables

$$I = \mu^2 P, \qquad W = Q, \qquad t = 4T$$

makes it possible to write the transformed Hamiltonian (4.3) in the form

$$H = \mu^2 H^{(0)}(P) + \mu^3 H^{(1)}(P, Q, T) + \ldots \tag{4.11}$$

For $0 < \mu < \mu^*$ function $H$ is analytic in the region

$$|\operatorname{Im} Q, \ T| \leqslant \rho, \qquad 1 - \delta < P < 2 + \delta$$

where $\mu^*$, $\rho$, $\delta$ are certain small numbers, and has a period of $2\pi$ with respect to $Q$ and $T$. The function $H^{(0)}$ in (4.11) is of the form

$$H^{(0)} = \frac{\pi^2 (b_2 + c_2)}{K^2(k)} P^2 \quad (4.12)$$

Since $d^2H^{(0)}/dP^2 \neq 0$, the neighborhood of $1 \leqslant P \leqslant 2$ is according to [3] filled for small $\mu$ to within a remainder of a small order of magnitude with invariant tori of the system with Hamiltonian (4.11). Consequently its trajectories, beginning sufficiently close to the coordinate origin, do not leave for all $t$ the neighborhood of $0 \leqslant P < 2$. Taking into account the relation between $P$ and initial variables $x, y$, we obtain the confirmation of stability of the equilibrium position $x = y = 0$.

We note that when the inequality $|c_2| > b_2$ is satisfied, there exists a power series (possible divergent) which formally is the fixed sign integral of system (1.1) [11]. In the problem considered here the Liapunov stability, according to Theorem 4.1, follows from the formal stability.

**5.** Let the Hamiltonian (1.2) be such that $\lambda k$ is not an integer for $k = 1, 2, \ldots, 2n$, and the coefficients $c_2, c_3, \ldots, c_n$ are zero. Then the question of stability is not determined by terms of order $2n$ in the expansion of the Hamiltonian.

Let us now assume that $\lambda(2n+1)$ is an integer; then the Hamiltonian (1.2) can be transformed into

$$H = ar^n \sqrt{r} \cos(2n+1)\varphi + O(r^{n+1}) \qquad (a = \text{const}) \quad (5.1)$$

With the use of the Liapunov function

$$V = r^n \sqrt{r} \sin(2n+1)\varphi \quad (5.2)$$

it is easy to prove that for $a \neq 0$ the equilibrium position is unstable.

Let furthermore either $a=0$, or $\lambda k$ not be an integer for $k = 1, 2, \ldots, 2n+1$, while $2\lambda(n+1)$ is an integer; then the Hamiltonian reduces to the form

$$H = r^{n+1}[c + b\cos 2(n+1)\varphi] + O(r^{n+3/2}) \quad (5.3)$$

where $c$ and $b$ are constant coefficients. For $|b| > |c|$ the equilibrium position is unstable. This is proved by using the Liapunov function

$$V = r^{n+1} \sin 2(n+1)\varphi \quad (5.4)$$

If, however, $|b| < |c|$, then according to Liapunov, the equilibrium position is stable. To prove this, we use the canonical transformation $r, \varphi \to I, w$ and the generating function

$$S(I, \varphi) = \frac{\pi I}{2(n+1)G_{n+1}} \int_0^{(n+1)\varphi} \frac{d\alpha}{\sqrt[n+1]{1 - k^2 \sin^2\alpha}} \qquad \left(k^2 = \frac{2b}{b+c}\right) \quad (5.5)$$

The signs of $b$ and $c$ can be assumed to be the same, hence $0 \leqslant k^2 < 1$. In (5.5) the following notation was used

$$G_{n+1} = \int_0^{\pi/2} \frac{d\alpha}{(1 - k^2\sin^2\alpha)^{1/(n+1)}}$$

In terms of new variables the Hamiltonian is of the form

$$H = \left(\frac{\pi I}{2G_{n+1}}\right)^{n+1}(b + c) + \Gamma(I, W, t)$$

where the function $\Gamma = O(I^{n+3/2})$ and is periodic with respect to $W$ and $t$.

Further proof is reduced to applying the results given in [3] as has been done in Sect. 4.

**6.** Let us now examine the stability of equilibrium position of the autonomous Hamiltonian system with two degrees of freedom. Let us assume that the frequencies of the linear approximation are not zero and are connected by the relation $\omega_1 = 3\omega_2$ and that the quadratic part in the expansion of the Hamiltonian is not of fixed sign. With a suitable choice of variables $q_j$, $p_j$ $(j = 1,2)$ the Hamiltonian can be written in the form

$$H = \omega_1 r_1 - \omega_2 r_2 + c_{20} r_1^2 + c_{11} r_1 r_2 + c_{02} r_2^2 + b r_2 \sqrt{r_1 r_2} \cos(\varphi_1 + 3\varphi_2) + \\ + H'(q_j, p_j) \tag{6.1}$$

where $H'$ is an analytic function of $q_j$, $p_j$

$$H' = O\left((r_1 + r_2)^{5/2}\right)$$

$$q_j = \sqrt{2r_j} \sin \varphi_j, \qquad p_j = \sqrt{2r_j} \cos \varphi_j$$

We introduce the notation

$$a_1 = c_{20} + 3c_{11} + 9c_{02}, \qquad b_1 = 3\sqrt{3}b$$

It is shown in [12, 13] that, if $|a_1| < |b_1|$, the equilibrium position is unstable, while in the case of $|a_1| > |b_1|$ a formal stability occurs. Let us show that the Liapunov stability follows from formal stability.

Using the integral $H = h = \text{const}$, we reduce the system with two degrees of freedom to a system with one degree of freedom, but with a $2\pi$-periodic dependence of the new Hamiltonian on the new independent variable. Since the motion is considered to be in a sufficiently small neighborhood of the coordinate origin, it can be assumed that $r_j \sim \varepsilon$ $(0 < \varepsilon \ll 1)$. Let the trajectory begin sufficiently close to the coordinate origin, so that $h \sim \varepsilon^{5/2}$, then, solving the equation $H = h$ for $r_2$, we obtain

$$r_2 = -\Phi_0 (r_1, \varphi_1, \varphi_2) - \Phi_1 (r_1, \varphi_1, \varphi_2, h)$$

where

$$\Phi_0 = -3r_1 - \omega_2^{-1} [a_1 + b_1 \cos(\varphi_1 + 3\varphi_2)] r_1^2$$

Function $\Phi_1 = O(r_1^{5/2})$ has a period of $2\pi$ with respect to $\varphi_1$ and to the new independent variable $\varphi_2$. If angle $\varphi = \varphi_1 + 3\varphi_2$ is substituted for $\varphi_1$ the Hamiltonian $\Phi$ of the obtained system with one degree of freedom is of the form

$$\Phi = -\omega_2^{-1} (a_1 + b_1 \cos \varphi) r_1^2 + R(r_1, \varphi, \varphi_2, h)$$

The signs of $a_1$ and $b_1$ can be assumed to be the same. Let us change the variables $r_1, \varphi \to I, W$ using the generating function

$$S = \frac{\pi I}{K(k)} F(\varphi/2, k) \left(k^2 = \frac{2b_1}{a_1 + b_1} < 1\right) \tag{6.2}$$

where $K$ and $F$ are elliptic integrals and $k$ is their modulus. The Hamiltonian $\Phi$ becomes

$$\Phi = -\frac{\pi^2 (a_1 + b_1)}{4\omega_2 K^2(k)} I^2 + R'(I, W, \varphi_2, h) \tag{6.3}$$

The function $R' = O(I^{5/2})$ has a period of $2\pi$ with respect to $W$ and $\varphi_2$ and is analytic with respect to all variables in the region

$$0 < \delta_1 \leqslant I \leqslant \delta_2, \quad |h| < \delta_3, \quad |\operatorname{Im} W, \varphi_2| < \delta_4$$

where $\delta_i$ are certain small real numbers. By applying to the system with Hamiltonian (6.3) the results of the analysis presented in [3] it can be readily shown that for every admissible value of $h$ in a sufficiently small neighborhood of the coordinate origin there exist invariant tori. From this follows the stability of the equilibrium position $q_j = p_j = 0$.

BIBLIOGRAPHY


1. Birkhoff, J. D., Dynamic Systems. M.-L., Gostekhizdat, 1941.
2. Moser, J., On curves invariant in the mapping of a torus which conserve the area. Sb. perew. i obz. in. period. lit., Matematika, Vol. 6, №5, p. 51.
3. Arnol'd, V. I., Small denominators and the problem of stability of motion in the classical and celestial mechanics. Usp. matem. n., Vol. 18, №6, 1963.
4. Levi-Civita, T., Sorpa alcuni criteri di instabilita. Ann. mat. pura et appl., Ser. 3, Vol. 5, p. 221, 1901.
5. Siegel, K. L., Lectures on Celestial Mechanics. (Russian translation). M., Izd. inostr. lit., 1959.
6. Merman, G. A., On the instability of the periodic solution of the canonical system with one degree of freedom in the case of main resonance. Sb. "The Problem of Motion of Artificial Celestial Bodies". M., Izd. Akad. Nauk SSSR, 1963.
7. Kamenkov, G. V., Investigation of stability of periodic motions. Tr. Univ. druzhby narodov im. Patrisa Lumumby. Teoret. mekhan., Vol. 15, №3, 1966.
8. Mustakhishev, K. M., On the questions of stability of Hamiltonian systems. Izv. Akad. Nauk KazSSR, Ser. Fiz.-matem., №1, 1967.
9. Liapunov, A. M., Sobr. Soch., Vol. 2, M., Izd. Akad. Nauk SSSR, 1956.
10. Landau, L. D. and Lifshits, E. M., Theoretical Physics, Vol. 1. Mekhanika, M., "Nauka", 1965.
11. Moser, J., Stabilitätsverhalten kanonischer Differentialgleichungssysteme. Nachr. Akad. Wiss. Göttingen, Math.-phys., Kl. IIa, №6, S. 87-120, 1955.
12. Markeev, A. P., Stability of a canonical system with two degrees of freedom in the presence of resonance. PMM Vol. 32, №4, 1968.
13. Markeev, A. P., On the stability of a nonautonomous Hamiltonian system with two degrees of freedom. PMM Vol. 33, №3, 1969.


Translated by J. J. D.

# AN ALTERNATIVE FOR THE GAME PROBLEM OF CONVERGENCE

PMM Vol. 34, №6, 1970, pp. 1005-1022

N. N. KRASOVSKII and A. I. SUBBOTIN
(Sverdlovsk)

(Received July 24, 1970)

In this paper a new class of generalized mixed strategies of players is presented, related to the problem of bringing a motion, under a control involving conflict, to a specified set under a phase restriction. This class of problems is so wide that it includes strategies which give saddle-point type situations in typical differential games. The contents of this paper are related to the problems discussed in [1-4] and the discussions are based on the extremal construction introduced in [5-7].

**1.** Consider first a motion under control involving conflict described by

$$dx/dt = f(t, x, u, v), \quad x[t_0] = x_0 \tag{1.1}$$

where $x$ is the $n$-dimensional phase vector of the system, $u$ and $v$ are the control force